

# NEAREST NEIGHBOR SPACING DISTRIBUTIONS FOR ZEROS OF THE REAL OR IMAGINARY PART OF THE RIEMANN XI-FUNCTION ON VERTICAL LINES

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ABSTRACT. We show that the density functions of nearest neighbor spacing distributions for zeros of the real or imaginary part of the Riemann xi-function on vertical lines are described by the  $M$ -function which is appeared in value distributions of the logarithmic derivative of the Riemann zeta-function on vertical lines.

## 1. INTRODUCTION

Let  $s = \sigma + it$  ( $i = \sqrt{-1}$ ) be a complex variable,  $\zeta(s)$  be the Riemann zeta-function, and

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

be the Riemann xi-function, which is an entire function satisfying functional equations  $\xi(s) = \xi(1-s)$  and  $\xi(\bar{s}) = \overline{\xi(s)}$ . In this paper, we discuss the distributions of zeros of entire functions

$$A_\omega(s) := \frac{1}{2}(\xi(s+\omega) + \xi(s-\omega)), \quad B_\omega(s) := \frac{i}{2}(\xi(s+\omega) - \xi(s-\omega)) \quad (1.1)$$

having a positive real parameter  $\omega$  in consideration of the following two relations with the zeros of  $\xi(s)$ . Firstly, the zeros of  $A_\omega(s)$  and  $B_\omega(s)$  on the line  $\sigma = 1/2$  coincide respectively with the zeros of the real and imaginary parts of  $\xi(s)$  on the line  $\sigma = 1/2 + \omega$ , because we have

$$\operatorname{Re} \xi\left(\frac{1}{2} + \omega + it\right) = A_\omega\left(\frac{1}{2} + it\right), \quad \operatorname{Im} \xi\left(\frac{1}{2} + \omega + it\right) = -B_\omega\left(\frac{1}{2} + it\right) \quad (1.2)$$

by functional equations of  $\xi(s)$ . Secondly, for small  $\omega > 0$ , the zeros of  $A_\omega(s)$  and  $B_\omega(s)$  (locally) approximate the zeros of  $\xi(s)$  and  $\xi'(s)$  respectively, because of asymptotic relations

$$A_\omega(s) = \xi(s) + O(\omega^2), \quad B_\omega(s) = i\omega \cdot \xi'(s) + O(\omega^3) \quad (\omega \rightarrow 0^+)$$

on compact subsets of  $\mathbb{C}$ .

The functional equations of  $\xi(s)$  deduce that  $A_\omega(s)$  and  $B_\omega(s)$  satisfy

$$A_\omega(s) = A_\omega(1-s), \quad B_\omega(s) = -B_\omega(1-s)$$

and take real values on the critical line  $\sigma = 1/2$ . It is known that all zeros of  $A_\omega(s)$  and  $B_\omega(s)$  are simple zeros lying on the critical line if  $\omega \geq 1/2$ . This holds also for  $0 < \omega < 1/2$  if we assume the Riemann Hypothesis (RH) for  $\xi(s)$  ([10, Theorem 2.1]), or unconditionally, except for a set of zeros up to height  $T$  of cardinality  $\ll T^{1-a\omega}(\log T)^2$  for any  $a < 1$  ([12, Theorem 1 and 2]). In this sense, the horizontal distributions of the zeros of  $A_\omega(s)$  and  $B_\omega(s)$  are understood well. Therefore we turn interest to their vertical distributions in what follows.

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Let  $X_\omega(s)$  be  $A_\omega(s)$  or  $B_\omega(s)$ . We arrange the zero  $\rho = \beta + i\gamma$  of  $X_\omega(s)$  with  $\gamma > 0$  in a sequence  $\rho_n = \beta_n + i\gamma_n$  so that  $\gamma_{n+1} \geq \gamma_n$ . Then the distribution of spacings of the normalized imaginary parts

$$\gamma_n^{(1)} := \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} \quad (1.3)$$

converges to a limiting distribution of equal spacings of length one. This fact is proved in Lagarias [10, Theorem 4.1] by assuming RH if  $0 < \omega < 1/2$  and in Li [12, Theorem 1] unconditionally. The above result on the normalized imaginary parts is contrast to the Montgomery–Odlyzko conjecture and the GUE conjecture which assert that the distribution of the normalized imaginary parts of the zeros of  $\xi(s)$  obeys the distribution of eigenvalues of random hermitian matrices from the Gaussian Unitary Ensemble (GUE). Therefore, one may consider that the zeros of  $A_\omega(s)$  and  $B_\omega(s)$  are insignificant objects at least from the viewpoint of their vertical distributions.

However, interestingly enough, it will be proved that the *second normalization* of the imaginary parts defined by

$$\gamma_n^{(2)} := \left( \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} - n \right) \varrho_\omega^{-1/2} \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} \quad (1.4)$$

have a remarkable distribution which is related to the Euler product of the Riemann zeta-function, where

$$\varrho_\omega := \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{1+2\omega}}$$

for the von Mangoldt function  $\Lambda(n)$  and the series converges absolutely for  $\omega > 0$ .

In order to state the main theorem, we recall a result on the value distributions of the logarithmic derivative of the Riemann zeta-function on vertical lines. For every  $\sigma > 1/2$ , there exists a non-negative real valued  $C^\infty$ -function  $M_\sigma(z)$  on  $\mathbb{C}$  such that  $(2\pi)^{-1} \int_{\mathbb{C}} M_\sigma(z) dz = 1$  and the formula

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \Phi \left( \frac{\zeta'}{\zeta}(\sigma + it) \right) dt = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(z) \Phi(z) dz \quad (1.5)$$

holds for any continuous bounded function  $\Phi(z)$  on  $\mathbb{C}$  or the characteristic function of either a compact subset of  $\mathbb{C}$  or the complement of such a subset. We call  $M_\sigma(z)$  the *M-function* according to [3]. The above formula was obtained by Kampen–Wintner [8], Kershner–Wintner [9], Guo [2], Ihara [3] and Ihara–Matsumoto [6] (see Appendix for a construction of  $M_\sigma(z)$  and its historical details). If  $\sigma > 1$ , formula (1.5) holds for any continuous function  $\Phi(z)$  on  $\mathbb{C}$ .

Using the *M-function*, we define the *m-function* by

$$m_\sigma(u) = \int_{-\infty}^{\infty} M_\sigma(u + iv) dv \quad (1.6)$$

on the real line. This is well-defined because  $M_\sigma(z)$  is of rapid decay ([3, Theorem 2]). Reflecting the Euler product formula of the Riemann zeta-function, the Fourier transform  $\tilde{M}_\sigma(z)$  has an Euler product formula  $\tilde{M}_\sigma(z) = \prod_p \tilde{M}_{\sigma,p}(z)$  whose local factors  $\tilde{M}_{\sigma,p}(z)$  are some arithmetic Dirichlet series in  $\sigma$ , where  $p$  runs over all prime numbers (see Appendix). Therefore, the Fourier transform of *m-function* also has an Euler product, since

$$\tilde{m}_\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} m_\sigma(u) e^{ixu} du = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(u + iv) e^{ixu} du dv = \tilde{M}_\sigma(x).$$

Now the main result is stated as follows.

**Theorem 1.** *Let  $X_\omega(s)$  be  $A_\omega(s)$  or  $B_\omega(s)$  for given  $\omega > 0$ , and let  $\gamma_n^{(2)}$  be the secondary normalized imaginary parts of the zeros of  $X_\omega(s)$  defined in (1.4). Then the formula*

$$\lim_{T \rightarrow \infty} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi(\gamma_{n+1}^{(2)} - \gamma_n^{(2)}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi \varrho_\omega^{1/2} m_{\frac{1}{2}+\omega}(\pi \varrho_\omega^{1/2} u) \phi(u) du \quad (1.7)$$

*holds for any bounded function  $\phi \in C^1(\mathbb{R})$  such that  $\phi'(x) \ll 1$  for  $|x| \leq 1$ ,  $\phi'(x) \ll x^{-2}$  for  $|x| \geq 1$  and  $u \mapsto \frac{d}{du} \phi(\operatorname{Re} \zeta'(\frac{1}{2} + \omega + iu))$  is bounded on  $\mathbb{R}$ , where  $N_\omega(T)$  is the number of zeros of  $X_\omega(s)$  with  $0 < t \leq T$ .*

The limit behavior of the integrand of the right-hand side of (1.7) as  $\omega \rightarrow 0^+$  is obtained as follows by using a result of [4].

**Theorem 2.** *We have*

$$\frac{1}{2\pi} \lim_{\omega \rightarrow 0^+} \pi \varrho_\omega^{1/2} m_{\frac{1}{2}+\omega}(\pi \varrho_\omega^{1/2} u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right).$$

Note that the above two theorems are unconditional.

We now make a consideration on a significance of Theorem 1 under RH if  $0 < \omega < 1/2$ . In this case, all zeros of  $X_\omega(s)$  are simple zeros lying on the critical line and

$$N_\omega(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + S_\omega(T) + \frac{7+2\omega}{8} + O\left(\frac{1}{T}\right), \quad (1.8)$$

for  $T \geq 2$  ([10, Theorem 3.1]), where

$$S_\omega(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + \omega + it\right)$$

is a  $C^\infty$ -function on the real line obtained by continuous variation along the straight lines joining  $2$ ,  $2 + it$  and  $1/2 + \omega + it$ , starting with the value  $0$ . By the simplicity of zeros, (1.3) and (1.8), we have

$$1 = N_\omega(\gamma_{n+1}) - N_\omega(\gamma_n) = \gamma_{n+1}^{(1)} - \gamma_n^{(1)} + S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) + O\left(\frac{1}{\gamma_n}\right),$$

and thus

$$\gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1 = -\left(S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)\right) + O\left(\frac{1}{\gamma_n}\right). \quad (1.9)$$

Given this formula,  $\gamma_{n+1}^{(1)} - \gamma_n^{(1)} \rightarrow 1$  means that the contribution of  $S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)$  is smaller than  $1$  for any fixed  $\omega > 0$ . In other words, the distribution of spacings of the normalized zeros of  $X_\omega(s)$  is dominated by the gamma functor of  $\zeta(s)$  only.

On the other hand, it is known that a subtle behavior of the zeros of  $\zeta(s)$  such as the Montgomery–Odlyzko conjecture is caused by the function  $S(t)$ , which is obtained by  $S(t) = \lim_{\omega \rightarrow 0^+} S_\omega(t)$  if  $t$  is not the ordinate of a zero of  $\zeta(s)$ , and  $S(t) = \frac{1}{2} \lim_{\delta \rightarrow 0^+} (S(t+\delta) + S(t-\delta))$  if  $t$  is not the ordinate of a zero of  $\zeta(s)$ .

Therefore, from the discussion above, Theorem 1 shows that the second normalization (1.4) detects an effect of the arithmetic part  $S_\omega(T)$  of the counting function  $N_\omega(T)$ . An Euler product formula of  $\tilde{m}_\sigma(u)$  is a supporting evidence of this observation.

A motivation of this work was L. Weng’s question to the author. In 2013, he and D. Zagier proved that all high-rank zeta functions for elliptic curves  $E$  defined over a finite field satisfy an analogue of the Riemann Hypothesis ([17]). Then he considered a distribution of the zeros of high-rank zeta functions for  $E$  when the rank is varied and observed that the dominant term is very simple but the second dominant term is related to the Sato-Tate measure. His question to the author was what an analogue of his observation to the number field case is ([15], where he considered another version of (1.4) but it is simplified in [16] as compatible with (1.4)). For the rational number field  $\mathbb{Q}$ , high-rank zeta functions  $\hat{\zeta}_{\mathbb{Q},n}(s)$  are expressed as linear combinations of products of

the Riemann zeta-function and rational functions. The rank one case is  $\hat{\zeta}_{\mathbb{Q},1}(s) = \hat{\zeta}(s)$ . The rank two case is

$$s(2s-1)(2s-2)\hat{\zeta}_{\mathbb{Q},2}(s) = \xi(2s) - \xi(2s-1) = B_{1/2}(2s - \tfrac{1}{2}).$$

Therefore, the second dominant term of the distribution of the zeros is described by  $m_1(x)$ . The rank three case is

$$\begin{aligned} 3s(3s-1)(3s-2)(3s-3)\hat{\zeta}_{\mathbb{Q},3}(s) &= X(s) + X(1-s), \\ X(s) &= \left(3(2\xi(2)-1)s - 4\xi(2) + 3\right)\xi(3s) - \xi(3s-1). \end{aligned}$$

This looks similar to  $A_1(3s-1) = \xi(3s) + \xi(3s-2)$  in a sense. Therefore, it is expected that the second dominant term of the distribution of the zeros is described by  $m_{3/2}(x)$  up to a small correction.

This paper is organized as follows. In Section 2, we prepare some lemmas necessary for the proof of Theorem 1. In Section 3, we prove Theorem 1 under RH at first for the simplicity of argument. Then we prove Theorem 1 unconditionally and prove Theorem 2. In Section 4, we several comments and remarks on subjects of the paper. Finally, we provide a review of construction, basic properties and history of the  $M$ -function as an appendix.

## 2. PRELIMINARIES

Let  $\omega > 0$ . We will assume RH if  $0 < \omega < 1/2$  throughout this section. Then the imaginary parts of  $A_\omega(s)$  and  $B_\omega(s)$  are enumerated as

$$\cdots < \gamma_{-1}(B_\omega) < \gamma_{-1}(A_\omega) < \gamma_0(B_\omega) = 0 < \gamma_1(A_\omega) < \gamma_1(B_\omega) < \gamma_2(A_\omega) < \gamma_2(B_\omega) < \cdots.$$

with  $\gamma_{-n}(A_\omega) = -\gamma_n(A_\omega)$  and  $\gamma_{-n}(B_\omega) = -\gamma_n(B_\omega)$  for  $n \geq 1$ . We denote by  $\gamma_n$  the  $n$ th imaginary part  $\gamma_n(A_\omega)$  or  $\gamma_n(B_\omega)$  when  $n \geq 1$ .

**Lemma 1.** *We have*

$$\gamma_n = \frac{2\pi n}{\log n} \left( 1 + O\left(\frac{\log \log n}{\log n}\right) \right), \quad (2.1)$$

$$\log \frac{\gamma_n}{2\pi} = \log n \left( 1 + O\left(\frac{\log \log n}{\log n}\right) \right). \quad (2.2)$$

*These formulas are unconditional.*

**Remark.** It is claimed that

$$\gamma_n = \frac{2\pi n}{\log n} \left( 1 + O\left(\frac{1}{\log n}\right) \right)$$

in [10, p.171] standing on (1.8) and  $S_\omega(t) = O(\log t)$ . However, the author do not know how to exclude the factor  $\log \log n$  from (2.1).

*Proof.* Suppose that  $\gamma_n = \gamma_n(A_\omega)$ . We have  $S_\omega(T) = O(\log T)$  unconditionally as well as [14, Theorem 9.4], where the implied constant does not depend on  $\omega$ . Therefore,

$$n = N_\omega(\gamma_n) = \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} \left( 1 + O\left(\frac{1}{\gamma_n}\right) \right)$$

by the simplicity of zeros. This implies

$$\log n = \log \frac{\gamma_n}{2\pi} \left( 1 + \frac{\log \log \frac{\gamma_n}{2\pi e}}{\log \frac{\gamma_n}{2\pi}} + O\left(\frac{1}{\gamma_n \log \gamma_n}\right) \right).$$

Taking the quotient of these two equalities,

$$\begin{aligned}
\frac{2\pi n}{\log n} &= \gamma_n \frac{\log \frac{\gamma_n}{2\pi} \left(1 + O\left(\frac{1}{\gamma_n}\right)\right) - \left(1 + O\left(\frac{1}{\gamma_n}\right)\right)}{\log \frac{\gamma_n}{2\pi} \left(1 + \frac{\log \log \frac{\gamma_n}{2\pi e}}{\log \frac{\gamma_n}{2\pi}} + O\left(\frac{1}{\gamma_n \log \gamma_n}\right)\right)} \\
&= \gamma_n \left( \frac{1 + O\left(\frac{1}{\gamma_n}\right)}{1 + \frac{\log \log \frac{\gamma_n}{2\pi e}}{\log \frac{\gamma_n}{2\pi}} + O\left(\frac{1}{\gamma_n \log \gamma_n}\right)} + O\left(\frac{1}{\log \gamma_n}\right) \right) \\
&= \gamma_n \left( 1 + O\left(\frac{\log \log \gamma_n}{\log \gamma_n}\right) \right).
\end{aligned}$$

Therefore,

$$\gamma_n = \frac{2\pi n}{\log n} \left( 1 + O\left(\frac{\log \log \gamma_n}{\log \gamma_n}\right) \right).$$

In particular,  $n/(\log n) \ll \gamma_n$  by  $\gamma_n \rightarrow \infty$ . Hence we obtain (2.1) by  $\log \log \gamma_n / \log \gamma_n \ll \log \log(n/(\log n)) / \log(n/(\log n)) \ll \log \log n / \log n$ . By (2.1), we have

$$\log \frac{\gamma_n}{2\pi} = \log n \left( 1 - \frac{\log \log n}{\log n} \right) \left( 1 + O\left(\frac{\log \log n}{\log n}\right) \right) = \log n \left( 1 + O\left(\frac{\log \log n}{\log n}\right) \right).$$

This is nothing but (2.2). The case of  $\gamma_n = \gamma_n(B_\omega)$  is proved in a similar way.  $\square$

**Lemma 2.** *The gaps  $\gamma_{n+1} - \gamma_n$  tend to 0 as  $n \rightarrow \infty$ .*

*Proof.* We show that  $S_\omega(t) = o(\log t)$  holds for any fixed  $\omega > 0$ , because it implies Lemma 2 by (1.8). We have

$$\log \zeta\left(\frac{1}{2} + \omega + it\right) \ll \begin{cases} 1 & \text{if } \omega > 1/2, \\ \log \log t & \text{if } \omega = 1/2, \\ \log \log \log t & \text{if } \omega = 1/2 \text{ under RH}, \\ \frac{(\log t)^{1-2\omega}}{\log \log t} & \text{if } 0 < \omega < 1/2 \text{ under RH}. \end{cases}$$

for large  $t > 0$ , where the first line is a consequence of the absolute convergence of the Dirichlet series of  $\log \zeta(s)$ , the second line is shown in [11, Theorem 6.7] and the other cases are shown in [14, Theorem 14.5, §14.33]. These estimates imply  $S_\omega(t) = o(\log t)$ , since  $S_\omega(t) \ll |\log \zeta(1/2 + \omega + it)|$ .  $\square$

**Lemma 3.** *We have*

$$\frac{S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)}{\gamma_{n+1} - \gamma_n} = O(E_{1,\omega}(\gamma_n)) \quad (2.3)$$

with

$$E_{1,\omega}(t) = \begin{cases} 1 & \text{if } \omega > 1/2, \\ \frac{\log t}{\log \log t} & \text{if } \omega = 1/2, \\ \log \log t & \text{if } \omega = 1/2 \text{ under RH}, \\ (\log t)^{1-2\omega} & \text{if } 0 < \omega < 1/2 \text{ under RH}. \end{cases} \quad (2.4)$$

*Proof.* We have  $\pi S'_\omega(t) = \operatorname{Re}(\zeta'/\zeta)(1/2 + \omega + it)$  by the definition of  $S_\omega(t)$ , since  $\zeta(s)$  has no zeros in  $\operatorname{Re}(s) > 1/2$  by RH. Therefore,

$$\pi \left| \frac{S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)}{\gamma_{n+1} - \gamma_n} \right| \leq \left| \operatorname{Re} \left\{ \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\gamma \right) \right\} \right| \leq \left| \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\gamma \right) \right|$$

for some  $\gamma_n < \xi < \gamma_{n+1}$  by Lemma 2 and the mean value theorem. On the right-hand side, we have

$$\frac{\zeta'}{\zeta}(\frac{1}{2} + \omega + i\xi) \ll E_{1,\omega}(\xi), \quad (2.5)$$

where the first line of (2.4) is a consequence of the absolute convergence of the Dirichlet series of  $(\zeta'/\zeta)(s)$ , the second line of (2.4) is shown in [14, (5.14.7)] and the other cases of (2.4) are shown in [14, §14.33]. These estimates imply (2.3), since  $\log \xi < \log \gamma_{n+1} = \log \gamma_n + O(\gamma_n^{-1})$  by Lemma 2.  $\square$

**Lemma 4.** *We have*

$$\frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} = 1 + O(E_{2,\omega}(\gamma_n)), \quad (2.6)$$

where  $E_{2,\omega}(t) = E_{1,\omega}(t)/\log t$  for the function  $E_{1,\omega}(t)$  of (2.4).

*Proof.* We have

$$N_\omega(t+h) - N_\omega(t) = \frac{h}{2\pi} \log \frac{t}{2\pi} + S_\omega(t+h) - S_\omega(t) + O\left(\frac{1}{t+1}\right)$$

for  $0 \leq h \leq 1$  and  $t \geq 2$  as well as the proof of [10, Theorem 4.1], where the implied constant does not depend on  $h$ . Applying this to  $t = \gamma_n$  and  $h = \gamma_{n+1} - \gamma_n$  together with Lemma 2, we get

$$1 = N_\omega(\gamma_{n+1}) - N_\omega(\gamma_n) = \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} + S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) + O\left(\frac{1}{\gamma_n}\right)$$

for large  $n$ . This implies

$$(\gamma_{n+1} - \gamma_n) \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} \left(1 + O\left(\frac{1}{\log \gamma_n} \left| \frac{S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n)}{\gamma_{n+1} - \gamma_n} \right| \right)\right) = 1 + O\left(\frac{1}{\gamma_n}\right).$$

Applying (2.3) to the left-hand side, we obtain (2.6).  $\square$

**Lemma 5.** *Assume that  $f(t)$  belongs to  $C^1(\mathbb{R})$  and  $f'(t)$  is bounded on  $\mathbb{R}$ . Then,*

$$\frac{1}{\gamma_N} \sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1} - \gamma_n) = \frac{1}{\gamma_N} \int_0^{\gamma_N} f(t) dt + O\left(\frac{1}{\log \gamma_N}\right) \quad (2.7)$$

holds for large  $N > 0$ .

*Proof.* We have

$$\frac{1}{\gamma_N} \sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1} - \gamma_n) = \frac{1}{\gamma_N} \int_{\gamma_1}^{\gamma_N} f(t) dt + \frac{1}{\gamma_N} \sum_{n=1}^{N-1} \int_{\gamma_n}^{\gamma_{n+1}} (f(\gamma_n) - f(t)) dt + O\left(\frac{1}{\gamma_N}\right).$$

The second sum on the right-hand side is estimated as

$$\begin{aligned} \left| \sum_{n=1}^{N-1} \int_{\gamma_n}^{\gamma_{n+1}} (f(\gamma_n) - f(t)) dt \right| &\leq \sum_{n=1}^{N-1} \max_{\gamma_n \leq \xi \leq \gamma_{n+1}} |f'(\xi)| \int_{\gamma_n}^{\gamma_{n+1}} (t - \gamma_n) dt \\ &\leq \frac{1}{2} \max_{\gamma_1 \leq t < \infty} |f'(t)| \sum_{n=1}^{N-1} (\gamma_{n+1} - \gamma_n)^2. \end{aligned}$$

Here the sum on the right-hand side is estimated as

$$\sum_{n=1}^{N-1} (\gamma_{n+1} - \gamma_n)^2 \ll \sum_{n=1}^{N-1} \frac{1}{\log \gamma_n},$$

since  $\gamma_{n+1} - \gamma_n \ll (\log \gamma_n)^{-1}$  by (2.6). Using the Stietjes integral and integration by parts, we have

$$\sum_{n=1}^{N-1} \frac{1}{\log \gamma_n} \ll \int_{\gamma_1}^{\gamma_N} \frac{dN_\omega(t)}{(\log t)^2} \ll \int_{\gamma_1}^{\gamma_N} \frac{dt}{\log t} \ll \frac{\gamma_N}{\log \gamma_N}.$$

Hence we obtain (2.7).  $\square$

### 3. PROOFS OF RESULTS

At first, we prove Theorem 1 assuming RH if  $0 < \omega < 1/2$  after preparing two propositions standing on results in the previous section.

**Proposition 1.** *Assume that  $f(t)$  belongs to  $C^1(\mathbb{R})$  and is bounded on  $\mathbb{R}$ . Then,*

$$\frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} f(\gamma_n) = \frac{1}{T} \int_0^T f(t) dt + O(E_{2,\omega}(T)) \quad (3.1)$$

holds for large  $T > 0$ , where  $E_{2,\omega}(t) = E_{1,\omega}(t)/\log t$  for the function  $E_{1,\omega}(t)$  of (2.4).

*Proof.* It is sufficient to show that the left-hand side of (2.7) is equal to the left-hand side of (3.1) up to a reasonable error terms. We have

$$\begin{aligned} \frac{1}{\gamma_N} \sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1} - \gamma_n) &= \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_N}{2\pi} \\ &= \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi} \\ &\quad + \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi} \left( \frac{\log \frac{\gamma_N}{2\pi}}{\log \frac{\gamma_n}{2\pi}} - 1 \right) \\ &= S_1 + S_2, \end{aligned}$$

say. First we consider  $S_1$ . We have

$$\left| S_1 - \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) \right| \ll \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} \left| \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi} - 1 \right|.$$

For the sum on the right-hand side,

$$\sum_{n=1}^{N-1} \left| \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi} - 1 \right| \ll \sum_{n=1}^{N-1} E_{2,\omega}(\gamma_n) \ll \int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t) dN_\omega(t)$$

by (2.6) and the Stietjes integral. Here

$$\int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t) dN_\omega(t) \ll \int_{\gamma_1}^{\gamma_N} E_{2,\omega}(t) (\log t) dt \ll \gamma_N \log \gamma_N E_{2,\omega}(\gamma_N)$$

by integration by parts. Hence

$$\left| S_1 - \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) \right| \ll E_{2,\omega}(\gamma_N).$$

Next we consider  $S_2$ . We have

$$|S_2| \ll \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} \left( \frac{\log \frac{\gamma_N}{2\pi}}{\log \frac{\gamma_n}{2\pi}} - 1 \right)$$

by (2.6). Using the partial summation for the sum on the right-hand side,

$$\begin{aligned} \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} \left( \frac{\log \frac{\gamma_N}{2\pi}}{\log \frac{\gamma_n}{2\pi}} - 1 \right) &= \frac{2\pi}{\gamma_N} \int_{\gamma_1}^{\gamma_N} \left( \sum_{0 < \gamma_n \leq x} 1 \right) \frac{1}{x(\log \frac{x}{2\pi})^2} dx + O\left(\frac{1}{\gamma_N}\right) \\ &\ll \frac{1}{\gamma_N} \int_{\gamma_1}^{\gamma_N} x \log x \cdot \frac{1}{x(\log x)^2} dx + O\left(\frac{1}{\gamma_N}\right) \ll \frac{1}{\log \gamma_N}. \end{aligned}$$

From the above argument, we obtain

$$\frac{1}{\gamma_N} \sum_{n=1}^{N-1} f(\gamma_n)(\gamma_{n+1} - \gamma_n) = \frac{1}{\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi}} \sum_{n=1}^{N-1} f(\gamma_n) + O(E_{2\omega}(\gamma_N)),$$

since  $(\log t)^{-1} \ll E_{2,\omega}(t)$  for every  $\omega > 0$ . Combining this with (2.7) and

$$\frac{\gamma_N}{2\pi} \log \frac{\gamma_N}{2\pi} = N_\omega(\gamma_N) \left( 1 + O\left(\frac{1}{\gamma_N}\right) \right),$$

we obtain (3.1) and complete the proof.  $\square$

**Proposition 2.** *Let  $\phi(x)$  be a function of  $C^1(\mathbb{R})$ . Assume that  $\phi'(x) \ll 1$  for  $|x| \leq 1$ ,  $\phi'(x) \ll x^{-2}$  for  $|x| \geq 1$  and  $u \mapsto \frac{d}{du} \phi(\operatorname{Re} \frac{\zeta'}{\zeta}(\frac{1}{2} + \omega + iu))$  is bounded on  $\mathbb{R}$ . We define*

$$\ddot{\gamma}_n = \varrho_\omega^{1/2} \gamma_n^{(2)} = \left( \frac{\gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} - n \right) \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e}. \quad (3.2)$$

Then

$$\begin{aligned} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) &= \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi \left( -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\gamma_n \right) \right) \\ &\quad + O\left( \frac{\log \log T}{\log T} \right) + O(E_{2,\omega}(T)) \end{aligned} \quad (3.3)$$

holds for large  $T > 0$ .

*Proof.* On the right-hand side of (1.9), we have

$$S_\omega(\gamma_{n+1}) - S_\omega(\gamma_n) = \frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) (\gamma_{n+1} - \gamma_n)$$

for some  $\xi_n \in (\gamma_n, \gamma_{n+1})$  by the mean value theorem. Therefore,

$$\begin{aligned} \left( \gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1 \right) \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} \\ = -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} + O\left( \frac{\log \gamma_n}{\gamma_n} \right) \end{aligned} \quad (3.4)$$

by (1.9). On the other hand, we have

$$\ddot{\gamma}_{n+1} - \ddot{\gamma}_n = \left( \gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1 \right) \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} + \left( \gamma_{n+1}^{(1)} - (n+1) \right) \frac{1}{2\pi} \left( \log \frac{\gamma_{n+1}}{2\pi e} - \log \frac{\gamma_n}{2\pi e} \right)$$

by definitions (1.3) and (3.2). The second term of the right-hand side is estimated as

$$\begin{aligned} \left( \gamma_{n+1}^{(1)} - (n+1) \right) \frac{1}{2\pi} \left( \log \frac{\gamma_{n+1}}{2\pi e} - \log \frac{\gamma_n}{2\pi e} \right) &= \left( \gamma_{n+1}^{(1)} - (n+1) \right) \frac{1}{2\pi} \log \left( 1 + \frac{\gamma_{n+1} - \gamma_n}{\gamma_n} \right) \\ &\ll n \frac{\log \log n}{\log n} \cdot \frac{\gamma_{n+1} - \gamma_n}{\gamma_n} \ll \gamma_n \log \log \gamma_n \cdot \frac{1}{\gamma_n \log \gamma_n} = \frac{\log \log \gamma_n}{\log \gamma_n} \end{aligned}$$

by (2.1), (2.2) and (2.6).

By the above argument, we get

$$\ddot{\gamma}_{n+1} - \ddot{\gamma}_n = \left( \gamma_{n+1}^{(1)} - \gamma_n^{(1)} - 1 \right) \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi e} + O\left( \frac{\log \log \gamma_n}{\log \gamma_n} \right). \quad (3.5)$$



Combining (3.4) and (3.5), we obtain

$$\ddot{\gamma}_{n+1} - \ddot{\gamma}_n = -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} + O\left( \frac{\log \log \gamma_n}{\log \gamma_n} \right) \quad (3.6)$$

for some  $\xi_n \in (\gamma_n, \gamma_{n+1})$ . Therefore,

$$\begin{aligned} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) &= \phi \left( -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} + O\left( \frac{\log \log \gamma_n}{\log \gamma_n} \right) \right) \\ &= \phi \left( -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) \frac{\gamma_{n+1} - \gamma_n}{2\pi} \log \frac{\gamma_n}{2\pi e} \right) + O\left( \frac{\log \log \gamma_n}{\log \gamma_n} \right) \\ &= \phi \left( -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\xi_n \right) \left( 1 + E_{2,\omega}(\gamma_n) \right) \right) + O\left( \frac{\log \log \gamma_n}{\log \gamma_n} \right) \end{aligned}$$

by the mean value theorem and (2.6), since  $\phi'(x)$  is bounded.

Now we take  $T_0 > 0$  so that the size of the error term  $O(E_{2,\omega}(t))$  of Lemma 4 is less than  $1/2$  for every  $t \geq T_0$ . We put  $r(t) = -\operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + it \right)$ ,  $I_1(T) = \{t \in [T_0, T] : |r(t)| \leq 2/3\}$  and  $I_2(T) = \{t \in [T_0, T] : |r(t)| > 2/3\}$  so that  $[T_0, T] = I_1(T) \cup I_2(T)$ .

If  $\gamma_n \geq T_0$  and  $\xi_n \in I_1(T)$ , we have

$$\begin{aligned} \phi \left( r(\xi_n)(1 + O(E_{2,\omega}(\gamma_n))) \right) - \phi(r(\xi_n)) &= \pm \int_{r(\xi_n)}^{r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))} \phi'(u) du \\ &\ll |r(\xi_n)| E_{2,\omega}(\gamma_n) \leq E_{2,\omega}(\gamma_n), \end{aligned}$$

since  $|r(\xi_n)| \leq 1$  and  $|r(\xi_n)(1 + O(E_{2,\omega}(t)))| \leq 1$ .

If  $\gamma_n \geq T_0$  and  $\xi_n \in I_2(T)$ , we have

$$\begin{aligned} \phi \left( r(\xi_n)(1 + O(E_{2,\omega}(\gamma_n))) \right) - \phi(r(\xi_n)) &= \pm \int_{r(\xi_n)}^{r(\xi_n)(1+O(E_{2,\omega}(\gamma_n)))} \phi'(u) du \\ &\ll \left| \frac{E_{2,\omega}(\gamma_n)}{r(\xi_n)(1 + O(E_{2,\omega}(\gamma_n)))} \right| \ll E_{2,\omega}(\gamma_n), \end{aligned}$$

since  $|r(\xi_n)(1 + O(E_{2,\omega}(\gamma_n)))| \geq 1/3$ . Therefore,

$$\phi \left( r(\xi_n)(1 + E_{2,\omega}(\gamma_n)) \right) = \phi(r(\xi_n)) + O(E_{2,\omega}(\gamma_n))$$

for every  $\gamma_n \geq T_0$  and  $\xi_n \in [T_0, T]$ . Moreover, we have

$$\phi \left( r(\xi_n)(1 + E_{2,\omega}(\gamma_n)) \right) = \phi(r(\gamma_n)) + O\left( \frac{1}{\log \gamma_n} \right) + O(E_{2,\omega}(\gamma_n))$$

by the mean value theorem, since  $\frac{d}{du} \phi(r(u))$  is bounded on  $\mathbb{R}$ ,  $\gamma_n < \xi_n < \gamma_{n+1}$  and  $\gamma_{n+1} - \gamma_n \ll (\log \gamma_n)^{-1}$ . Therefore,

$$\begin{aligned} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) &= \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi \left( -\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \omega + i\gamma_n \right) \right) \\ &\quad + O\left( \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \frac{\log \log \gamma_n}{\log \gamma_n} \right) + O\left( \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} E_{2,\omega}(\gamma_n) \right). \end{aligned}$$

By the Stieltjes integral and integration by parts, we have

$$\sum_{0 < \gamma_n \leq T} \frac{\log \log \gamma_n}{\log \gamma_n} = \int_{\gamma_1}^T \frac{\log \log t}{\log t} dN_\omega(t) \ll \int_{\gamma_1}^T \frac{\log \log t}{\log t} (\log t) dt \ll T \log \log T$$

and

$$\sum_{0 < \gamma_n \leq T} E_{2,\omega}(\gamma_n) = \int_{\gamma_1}^T E_{2,\omega}(t) dN_\omega(t) \ll \int_{\gamma_1}^T E_{2,\omega}(t) (\log t) dt \ll N_\omega(T) E_{2,\omega}(T).$$

Hence we obtain (3.3).  $\square$

**3.1. Proof of Theorem 1 under RH.** Put  $\sigma = 1/2 + \omega$ . By Proposition 1 and 2,

$$\begin{aligned} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) &= \frac{1}{2T} \int_{-T}^T \phi\left(-\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it)\right) dt \\ &+ O\left(\frac{\log \log T}{\log T}\right) + O(E_{2,\omega}(T)) \end{aligned} \quad (3.7)$$

holds for large  $T > 0$ , since  $\operatorname{Re}(\zeta'/\zeta)(\sigma + it)$  is an even function of  $t \in \mathbb{R}$ .

For any continuous and bounded function  $\phi(x)$  on  $\mathbb{R}$ ,  $\phi(\operatorname{Re}(z))$  is a continuous and bounded function on  $\mathbb{C}$ , because  $z \mapsto \frac{1}{2}(z + \bar{z})$  is a continuous function from  $\mathbb{C}$  into  $\mathbb{R}$ . Therefore, by applying formula (1.5) to  $\Phi(z) = \phi(-\frac{1}{\pi} \operatorname{Re}(z))$ , we have

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \phi\left(-\frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it)\right) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi m_\sigma(\pi u) \phi(u) du,$$

since the  $m$ -function  $m_\sigma(u)$  of (1.6) is even. Hence we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} \phi(\ddot{\gamma}_{n+1} - \ddot{\gamma}_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi m_\sigma(\pi u) \phi(u) du,$$

since  $\lim_{T \rightarrow \infty} E_{2,\omega}(T) = 0$  for any fixed  $\omega > 0$ . This implies (1.7) by  $\gamma_n^{(2)} = \varrho_\omega^{-1/2} \ddot{\gamma}_n$ .  $\square$

**3.2. Proof of Theorem 1.** Let  $X_\omega(s)$  be  $A_\omega(s)$  or  $B_\omega(s)$ . We arrange the zero  $\rho = \beta + i\gamma$  of  $X_\omega(s)$  with  $\gamma > 0$  in a sequence  $\rho_n = \beta_n + i\gamma_n$  so that  $\gamma_{n+1} \geq \gamma_n$ . Firstly, we recall that the numbers of zeros of  $X_\omega(s)$  up to height  $T$  and outside the line  $\sigma = 1/2$  are bounded by  $T^{1-a\omega}(\log T)^2$  for any  $a < 1$  ([12, Theorem 1]). In addition, for given  $0 < \delta < 1$  and  $B > 0$ , we can take an open subset  $E \subset (0, \infty)$  such that

- the measure of  $[T, 2T] \cap E$  is bounded by  $T/(\log T)^B$  for every  $T \geq 2$ ,
- the number of zeros of  $X_\omega(1/2 + it)$  for  $t \in [T, 2T]$  is bounded by  $T/(\log T)^B$  for every  $T \geq 2$ ,
- the zeros of  $X_\omega(1/2 + it)$  for  $t \in [T, 2T] \setminus E$  are simple,
- $[\gamma_n, \gamma_{n+1}] \subset [T, 2T] \setminus E$  if  $\gamma_n \in [T, 2T] \setminus E$ ,
- $\gamma_{n+1} - \gamma_n = O(1/\log T)$  if  $\gamma_n \in [T, 2T] \setminus E$ ,
- $S_\omega(t)$  is of  $C^\infty$  class in  $(0, \infty) \setminus E$ ,
- the estimate

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2} + \omega + it\right) \ll (\log T)^{1-\delta} \quad (3.8)$$

holds for  $t \in [T, 2T] \setminus E$ ,

by [12, Theorem 1] and the proof of [12, Theorem 2]. Therefore, we have

$$\lim_{T \rightarrow \infty} \frac{1}{N_\omega(T)} \sum_{0 < \gamma_n \leq T} f(\gamma_n) = \lim_{T \rightarrow \infty} \frac{1}{N_\omega(T)} \sum_{\substack{0 < \gamma_n \leq T \\ \gamma_n \notin E}} f(\gamma_n).$$

Using (3.8) instead of (2.5) for a calculation of the right-hand side, we obtain (3.1), (3.3) and (3.7) by replacing  $E_{2,\omega}(t)$  by  $(\log t)^{-\delta}$  in a way similar to the conditional proof of Theorem 1. Hence we obtain Theorem 1.  $\square$

**3.3. Proof of Theorem 2.** Let  $\mu_\sigma$  be the variance of  $M_\sigma(z)$ :

$$\mu_\sigma = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(z) |z|^2 dudv. \quad (3.9)$$

Then we have

$$\varrho_\omega = \frac{1}{2\pi^2} \mu_\sigma$$

for  $\sigma = 1/2 + \omega$  by [5, (4.1.8), (4.2.1)] or [4, (1.2.17), (1.2.21)]. Thus, by using the Fourier inversion formula

$$M_\sigma(u + iv) = \frac{1}{2\pi} \int_{\mathbb{C}} \tilde{M}_\sigma(x + iy) e^{-i(xu + yv)} dx dy,$$

we obtain

$$\begin{aligned} \lim_{\omega \rightarrow 0^+} \pi \varrho_\omega^{1/2} m_{\frac{1}{2} + \omega}(\pi \varrho_\omega^{1/2} u) &= \lim_{\omega \rightarrow 0^+} \frac{1}{2} \int_{-\infty}^{\infty} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} \frac{u + iv}{\sqrt{2}}) dv \\ &= \lim_{\omega \rightarrow 0^+} \int_{-\infty}^{\infty} \tilde{M}_\sigma(\sqrt{2} \mu_\sigma^{-1/2} x) e^{-ixu} dx. \end{aligned}$$

The integrand of the right-hand side is estimated as

$$|\tilde{M}_\sigma(\sqrt{2} \mu_\sigma^{-1/2} z)| \leq \exp(-\sqrt{2}|z|/8)$$

as well as [4, (2.4.2)] if  $\sigma$  is sufficiently close to  $1/2$ . Therefore, by applying Lebesgue's convergence theorem to the right-hand side together with

$$\lim_{\sigma \rightarrow 1/2} \tilde{M}_\sigma(\mu_\sigma^{-1/2} z) = \exp(-|z|^2/4),$$

which is a special case of [4, Lemma A], we obtain

$$\lim_{\sigma \rightarrow 1/2} \int_{-\infty}^{\infty} \tilde{M}_\sigma(\sqrt{2} \mu_\sigma^{-1/2} x) e^{-ixu} dx = \int_{-\infty}^{\infty} \exp(-x^2/2) e^{-ixu} dx = \sqrt{2\pi} \exp\left(-\frac{u^2}{2}\right).$$

This implies Theorem 2.  $\square$

#### 4. CONCLUDING REMARKS

Before concluding the main parts of the paper, we give several comments and remarks.

**4.1. On the range of test functions.** In order to extend the range of test functions of formula (1.7), we need to extend the range of test functions of formula (1.5). An optimistic expectation is that formula (1.5) holds for any continuous function  $\Phi(z)$  on  $\mathbb{C}$  or the characteristic function of either a compact subset of  $\mathbb{C}$  or the complement of such a subset if we assume RH. However, the range of test functions of (1.5) could possibly be much more delicate problem. In fact, if we apply (1.5) formally to the test function  $\Phi(w) = |w|^2$  together with (A.3) below, we obtain

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| \frac{\zeta'}{\zeta}(\sigma + it) \right|^2 dt = \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}} = \mu_\sigma.$$

This agree with the asymptotic formula

$$\frac{1}{T} \int_0^T \left| \frac{\zeta'}{\zeta}(\sigma + it) \right|^2 dt \sim \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}}$$

for  $(\sigma - 1/2) \log T \rightarrow \infty$  which is followed from the estimate  $S(T) = O(\log T / \log \log T)$  of Selberg [13, (1.2)], where  $f \sim g$  means that the ratio  $f/g$  tends to one. It is easy to see that  $\mu_\sigma \sim 1/(2\sigma - 1)^2$  as  $\sigma \rightarrow 1/2$ . Thus, we obtain the asymptotic formula

$$\frac{1}{T} \int_0^T \left| \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt \sim \frac{1}{4a^2} (\log T)^2$$

as  $a \rightarrow \infty$  and  $T \rightarrow \infty$  with  $a = o(\log T)$ . On the other hand, Goldston–Gonek–Montgomery [1] discovered that, assuming RH,

$$\frac{1}{T} \int_0^T \left| \frac{\zeta'}{\zeta} \left( \frac{1}{2} + \frac{a}{\log T} + it \right) \right|^2 dt \sim \frac{1 - e^{-2a}}{4a^2} (\log T)^2$$

as  $T \rightarrow \infty$  for any fixed  $a > 0$  is equivalent to Montgomery–Odlyzko conjecture. The above facts do not contradict each other, but they suggest a need of careful consideration for the range of test functions when  $\sigma$  close to  $1/2$ .

**4.2. On the second normalization.** Applying (1.5) formally to the test function  $\Phi(w) = |\operatorname{Re}(w)|^2 = w^2 + 2w\bar{w} + \bar{w}^2$  together with (A.3) below, we have

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + it) \right|^2 dt = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n^{2\sigma}}.$$

Therefore, by (2.1), (2.2) and (3.6), we obtain

$$|\gamma_{n+1}'' - \gamma_n''| \approx \left| \frac{1}{\pi} \operatorname{Re} \frac{\zeta'}{\zeta}(\sigma + i\xi_n) \right| \approx \varrho_{\sigma-1/2}$$

on average in spite of (2.5). This is a reason on the normalizing factor  $\varrho_{\omega}^{-1/2}$  of (1.4). The factor  $(1/(2\pi)) \log(\gamma_n/(2\pi e))$  of (1.4) is a kind of technical adjustment to establish a bridge between the nearest-neighbour spacing of normalized zeros and the  $M$ -function.

**4.3. On a relation with Montgomery–Odlyzko conjecture.** The functions  $A_{\omega}(s)$  and  $B_{\omega}(s)$  are holomorphic in  $(\omega, s)$  as a function of two complex variables, and all their zeros are simple under RH if  $\omega$  is a nonzero real number. Hence the sets of imaginary parts of  $n$ th zeros  $\{\gamma_n(\omega) | \omega > 0\}$  make analytic loci in  $(0, \infty) \times (0, \infty)$ , and they do not intersect each other. Moreover, assuming the simplicity of zeros of  $\xi(s)$ ,  $\lim_{\omega \rightarrow 0} \gamma_{n+1}(\omega) \neq \lim_{\omega \rightarrow 0} \gamma_n(\omega)$  for each  $n \geq 1$ . Therefore, we expect that the distribution of  $\gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega)$  approximates well the distribution of the nearest-neighbor spacings  $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0)$  if  $\omega > 0$  is small enough. In this sense, the distribution of  $\gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega)$  should approximate the distribution of  $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$  up to a correction factor, since

$$\gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega) \sim \left( \gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega) - 1 \right) \frac{1}{2\pi} \log \frac{\gamma_n(\omega)}{2\pi e}$$

for large  $n$  by (3.5). Moreover, we have

$$\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega) \sim \left( \gamma_{n+1}^{(1)}(\omega) - \gamma_n^{(1)}(\omega) - 1 \right)$$

when  $\sqrt{2}\omega \log \gamma_n(\omega) \sim 1$  as  $\omega \rightarrow 0^+$ , since  $\rho_{\omega} \sim 1/(8\pi^2\omega^2)$  as  $\omega \rightarrow 0^+$ . Therefore, for small  $\omega > 0$ , the distribution of  $\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega)$  around the height  $\exp(1/\omega)$  approximate the  $-1$  shift of the nearest-neighbor spacing distribution of  $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$  in the same range. Conversely, the distribution of  $\gamma_{n+1}^{(1)}(0) - \gamma_n^{(1)}(0) - 1$  around a height  $T > 0$  is approximated by the distribution of  $\gamma_{n+1}^{(2)}(\omega) - \gamma_n^{(2)}(\omega)$  for  $\omega \sim 1/(\sqrt{2} \log T)$ .

However, the limit of the density function in Theorem 2 is quite different from a shift of the density function

$$p(u) \approx \frac{32}{\pi^2} u^2 \exp\left(-\frac{4}{\pi} u^2\right)$$

of the nearest-neighbour spacing distribution for GUE predicted in the Montgomery–Odlyzko conjecture. In order to fill this gap, we may need a detailed study of the second error term of (3.7), which tends to  $O(1)$  as  $\omega \rightarrow 0^+$ , and the effect of the normalizing factor  $\varrho_{\omega}$  of (1.4).

**4.4. On possible generalization.** Let  $L(s, f)$  be a self-dual  $L$ -function in a sense of Iwaniec–Kowalski [7, Chap. 5] which includes Dedekind zeta-functions, Dirichlet  $L$ -functions associated to real primitive characters, Hecke  $L$ -functions associated to self-dual Hecke characters, automorphic  $L$ -functions associated to self-dual primitive holomorphic/Maass cusp forms, etc. For such  $L$ -function, a family of functions  $A_\omega(s, f)$  and  $B_\omega(s, f)$  corresponding to (1.1) is defined as well, and it is established in a way similar to [10] that the distribution of spacings of the normalized imaginary parts of the zeros of  $A_\omega(s, f)$  and  $B_\omega(s, f)$  converges to a limiting distribution of equal spacings of length one if we assume the Grand Riemann Hypothesis and the Ramanujan–Petersson conjecture for  $L(s, f)$ . A key ingredient is an analogue of (2.5) and other standard analytic properties of  $L$ -functions (see [7, Chap. 5]). Therefore, an analogue of the second normalization (1.4) is defined as well.

However, an analogue of the  $M$ -function  $M_\sigma(z)$  is not known except for the case of Dedekind zeta functions. It is an interesting problem to find an analogue of the function  $M_\sigma(z)$  for  $L(s, f)$ , but it is not obvious what it is, even if it may not be hard to find an analogue of  $M_\sigma(z)$  by a way similar to [3] for degree one  $L$ -functions like Dirichlet/Hecke  $L$ -functions for real/self-dual characters.

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#### APPENDIX A. $M$ -FUNCTION

In this part, we review a construction and basic properties of the  $M$ -function  $M_\sigma(z)$  in formula (1.5) according to Ihara [3, 4] and Ihara–Matsumoto [5]. See these references for details.

Let  $\Lambda : \mathbb{N} \rightarrow \mathbb{R}$  be the von Mangoldt function, that is,  $\Lambda(n) = \log p$  if  $n = p^k$  for some prime number  $p$  and integer  $k \geq 1$ , and  $\Lambda(n) = 0$  otherwise. We define arithmetic functions  $\Lambda_k : \mathbb{N} \rightarrow \mathbb{R}$  by

$$\left(-\frac{\zeta'}{\zeta}(s)\right)^k = \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right)^k = \sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s}$$

for  $k \geq 1$  and  $\Lambda_0(n) = 1$  if  $n = 1$ , and  $\Lambda_0(n) = 0$  otherwise. For a positive integer  $n$  and  $z \in \mathbb{C}$ , we define

$$\lambda_z(n) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(n)}{k!} z^k.$$

The series converges absolutely and uniformly on every compact subset of  $\mathbb{C}$ , and it is a polynomial of  $z$  by [3, (3.8.5), (3.8.6)]. Moreover, we have

$$\lambda_z(mn) = \lambda_z(m)\lambda_z(n) \quad \text{if } (m, n) = 1 \tag{A.1}$$

([3, Prop. 3.8.11(i)]). For a prime number  $p$  and complex numbers  $s, z \in \mathbb{C}$ , we define

$$\tilde{M}_{s,p}(z) = \sum_{j=0}^{\infty} \frac{\lambda_{p^j}(z)\lambda_{p^j}(\bar{z})}{p^{2js}}$$

The series converges absolutely for all  $s$  with  $\operatorname{Re}(s) > 0$  and  $z$  in a compact subset of  $\mathbb{C}$  by [3, prop. 3.9.4(i)]. Using  $\tilde{M}_{s,p}(z)$ , we define  $\tilde{M}_s(z)$  by the Euler product

$$\tilde{M}_s(z) = \prod_p \tilde{M}_{s,p}(z), \tag{A.2}$$

where  $p$  runs over all prime numbers. The product converges for all  $s$  with  $\operatorname{Re}(s) > 1/2$  and  $z$  in a compact subset of  $\mathbb{C}$  ([3, Theorem 5]). We have the Dirichlet series expansion

$$\tilde{M}_s(z) = \sum_{n=1}^{\infty} \frac{\lambda_z(n)\lambda_{\bar{z}}(n)}{n^{2s}}$$

by (A.1) and the series on the right-hand side converges absolutely all  $s$  with  $\operatorname{Re}(s) > 1/2$  and  $z$  in a compact subset of  $\mathbb{C}$  by [3, prop. 3.9.4(ii)].

For  $\sigma > 1/2$  and  $z \in \mathbb{C}$ ,  $\tilde{M}_\sigma(z)$  is a real analytic function of  $\sigma$  and  $z$  which does not vanish identically, and satisfy  $\tilde{M}_\sigma(z) = \tilde{M}_\sigma(\bar{z}) = \overline{\tilde{M}_\sigma(-\bar{z})}$  and  $\tilde{M}_\sigma(z) = O((1 + |z|)^{-n})$  for any  $n \geq 1$ . The  $M$ -function in formula (1.5) is defined by the Fourier transform

$$M_\sigma(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \tilde{M}_\sigma(w) \psi_{-z}(w) dw,$$

where  $\psi_z(w) = \exp(i \cdot \operatorname{Re}(\bar{z}w))$ . In addition, the  $M$ -function is real valued, decays rapidly as  $|z| \rightarrow \infty$ , and the Fourier inversion formula

$$\tilde{M}_\sigma(z) = \frac{1}{2\pi} \int_{\mathbb{C}} M_\sigma(w) \psi_z(w) dw$$

holds with  $\tilde{M}_\sigma(0) = 1$  ([3, Theorem 2 and 3, Remark 3.4.6]). In particular,  $(2\pi)^{-1} M_\sigma(w) dw$  is a probabilistic measure on  $\mathbb{C}$ . Corresponding to the Euler product (A.2), the  $M$ -function has a convolution Euler product whose  $p$ -factor being a certain distribution.

We have

$$\frac{1}{2\pi} \int_{\mathbb{C}} w^a \bar{w}^b M_\sigma(w) dw = \sum_{n=1}^{\infty} \frac{\Lambda_a(n) \Lambda_b(n)}{n^{2\sigma}} \quad (\text{A.3})$$

unconditionally together with the absolute convergence of the series if  $\sigma > 1$  ([3, Theorem 6]). Moreover, we have the limit formula

$$\lim_{\sigma \rightarrow 1/2} \mu_\sigma M_\sigma(\mu_\sigma^{1/2} z) = 2e^{-|z|^2}$$

and the convergence is uniform on  $|z| \leq R$  for any  $R > 0$ , where  $\mu_\sigma$  is the variance in (3.9) ([4, Theorem 2]). Theorem 2 is a formal consequence of this formula.

Historically, formula (1.5) was obtained first in 1936 by Kershner–Wintner [9] for  $\sigma > 1/2$  in terms of asymptotic distribution functions as an analogue of a work of Jessen–Wintner for  $\log \zeta(s)$  in 1935. However, they did not explicitly give the density function. The density function  $M_\sigma(z)$  was constructed in 1937 by Kampen–Wintner [8] for  $\sigma > 1$  as an infinite convolution Euler product. After that formula (1.5) was rediscovered by Guo [2] in 1993. He constructed  $M_\sigma(z)$  for  $\sigma > 1/2$  as the Fourier transform of the Euler product  $\prod_p \tilde{M}_{\sigma,p}(z)$  but test functions in (1.5) are restricted to smooth and compactly supported functions. This restriction for the test functions was relaxed to a wider class of functions by Ihara–Matsumoto [6] in 2011 which was a goal of a series of collaboration works of Ihara and Matsumoto standing on Ihara [3]. In 2008, Ihara [3] studied analytic and arithmetic properties of  $M_\sigma(z)$  and  $\tilde{M}_\sigma(s)$  systematically and in detail for  $\sigma > 1/2$  motivated by a study on Euler–Kronecker constants of global fields. This work was refined in Ihara [4]. The formulation of (1.5) in the introduction depends on [3, Theorem 6] and [6].

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